Reduction of Jacobi manifolds

1997 J. Phys. A: Math. Gen. 30 2783

(http://iopscience.iop.org/0305-4470/30/8/022)

J. Phys. A: Math. Gen. **30** (1997) 2783–2798. Printed in the UK PII: S0305-4470(97)74554-2

Reduction of Jacobi manifolds

A Ibort†, M de León‡ and G Marmo§

† Departamento de Física Teórica, Universidad Complutense de Madrid, 28040 Madrid, Spain
‡ Instituto de Matemáticas y Física Fundamental, CSIC, Serrano 123, 28006 Madrid, Spain
§ Dipartimento di Scienze Fisiche, INFN Sezione di Napoli, Università di Napoli, Mostra d'Oltremare, Pad. 19, 80125 Napoli, Italy

Received 14 May 1996, in final form 1 October 1996

Abstract. A reduction procedure for Jacobi manifolds is described in the algebraic setting of Jacobi algebras. As applications, reduction by arbitrary submanifolds, distributions and the reduction of Jacobi manifolds with symmetry are discussed. This generalized reduction procedure extends the well known reduction procedures for symplectic, Poisson, contact and co-symplectic structures.

1. Introduction

Jacobi manifolds were introduced by A Lichnerowicz as a rich geometrical notion extending several important geometrical structures, including among others, symplectic, Poisson, contact and co-symplectic [Li78]. However, it is true that conceptually Jacobi manifolds arise from the notion of local Lie algebras, i.e. Lie algebras on spaces of smooth sections of vector bundles with a locality property [Sh74], [Ki76], [Gu84].

In this paper we analyse a geometrical reduction procedure for Jacobi manifolds. It is well known that reduction is a natural geometrical process in the categories of Poisson, symplectic, contact and co-symplectic manifolds [Ma86], [Ma74], [Al89] and also in the category of Poisson supermanifolds [Ca90]. It is thus natural to ask if there is a natural mechanism for the reduction of Jacobi manifolds. It so happens that the adequate way to address such a generalization is using an algebraic formulation as proposed in [Gr94] or [Ki93] for Poisson manifolds and in [Ma96] for Nambu manifolds. The algebraic approach benefits from the natural duality between algebras of functions and sets. We will use such an approach here and show that there is a natural generalization of this algebraic reduction procedure to the category of Jacobi manifolds and Jacobi algebras [Gr92].

This mechanism allows us to construct new Jacobi manifolds out of simple ones very much like symplectic reduction offers a simple way to construct complicated (from the topological point of view) symplectic manifolds from 'simple' symplectic manifolds. As a byproduct of these ideas we will be able to offer another interpretation of several theorems concerning the existence of Jacobi structures on certain submanifolds and projections of Jacobi manifolds [Ju84], [Da91].

There is a striking similarity between Jacobi manifolds and certain algebraic structures that have arisen recently in quantum field theories called BV-algebras and some of their generalizations [Wi90], [Ge94], [Sc95]. In a sense that will be discussed in forthcoming papers, the Jacobi algebras are one instance of a classical limit of a generalized notion of

0305-4470/97/082783+16$19.50

BV-algebras. This remark raises the problem of quantizing Jacobi manifolds as a question of physical interest.

It is worth pointing out that the algebraic approach to the reduction of geometric structures focuses the attention on the way to reproduce the initial geometrical structures on algebras constructed from the initial one using simple algebraic constructions such as quotienting out by ideals and restricting to suitable subalgebras. These constitute in fact the two elementary steps in the reduction process. This is very much a reminder of the procedure of 'localization' in algebraic geometry even though the outcome is very different.

The paper is organized as follows. We will review succintly some of the main features of Jacobi manifolds that will be used in the paper in section 2, in particular we will review the characterization of Jacobi algebras and the role of derivations of the algebra of smooth functions. In section 3 we will describe the algebraic reduction procedure for associative commutative algebras and for associative commutative algebras with a Lie algebra structure (but not necessarily Poisson algebras). Then we will discuss the reduction of Jacobi algebras, i.e. algebras of functions of Jacobi manifolds with respect to an ideal, a submanifold and a distribution, and in section 4 we will compare these results with the well known reduction procedures in symplectic, Poisson etc, manifolds.

2. Jacobi manifolds and Jacobi algebras

2.1. Geometric and algebraic characterization of Jacobi manifolds

Definition 1. A Jacobi manifold [Li78] is a triple (M, Λ, X) where M is a smooth manifold, Λ is a bivector and X is a vector field defined on M, such that they satisfy the relations:

(i) $[\Lambda, \Lambda] = 2X \wedge \Lambda$,

(ii) $[X, \Lambda] = 0$.

In the previous definition $[.,.]$ denotes the Schouten bracket in the exterior algebra of multivectors on the manifold M [Sc54], [Tu74].

We will denote by $\mathcal{F} = \mathcal{F}(M)$ the algebra of smooth functions on the manifold M. A Lie algebra structure $[.,.]$ on $\mathcal{F}$ will be said to be local if the linear operator D_f on $\mathcal{F}$ defined by

$$D_f(g) = [f, g] \qquad \forall g \in \mathcal{F} \tag{1}$$

is local for all $f \in \mathcal{F}$. It is an important result that Jacobi manifolds are distinguished by the following fact [Sh74], [Ki76], [Gu84].

Theorem 1. Any local Lie algebra structure $[.,.]$ on the algebra $\mathcal{F}$ of smooth functions of a manifold M is associated with a pair of tensors Λ, X satisfying the conditions (i) and (ii) in definition 1, and such that the bracket of two functions f, g has the form

$$[f, g] = \{f, g\}_\Lambda + f\mathcal{L}_X g - g\mathcal{L}_X f \tag{2}$$

where $\{f, g\}_\Lambda$ denotes the bracket defined by the bivector Λ given by

$$\{f, g\}_\Lambda = \Lambda(\mathrm{d}f, \mathrm{d}g)$$

and $\mathcal{L}_X f$ denotes the Lie derivative of f along the vector field X.

The same result holds true for local Nambu structures [Mr96] replacing Λ by an N-multivector and X by a $(N-1)$ multivector satisfying similar compatibility conditions to (i) and (ii) in definition 1. It is worth pointing out that $(\mathcal{F}, [.,.], \cdot)$ is not a Poisson algebra since

$$[f, gh] = g[f, h] + [f, g]h + gh(\mathcal{L}_X f) \tag{3}$$

i.e. the bracket $[.,.]$ is not a derivation on each factor because of the last term in the right-hand side of (3). However, the bracket $[.,.]$ is not far from being a derivation because it defines a differential operator of order one. Let us recall that a differential operator of order r on an associative commutative algebra A is defined recursively as a linear map D on A such that $\delta(x)D$ is a differential operator of order $r-1$ for all $x \in A$, where $[\delta(x)D](y) = D(xy) - xD(y)$. Differential operators of order zero are defined as multiplication by elements on the algebra. It is not hard to see that a linear operator D on A is a differential operator of order r iff $\delta(x)^{r+1}D = 0$ for all $x \in A$. In this sense the linear operator D_f on $\mathcal{F}$ defined by (1) is a differential operator of order one for all $f \in \mathcal{F}$. In fact, a simple computation using (3) shows that $\delta(g)^2 D_f = 0$ for all $g \in \mathcal{F}$. The skew-symmetry of the bracket $[.,.]$ guarantees that it is also a differential operator on the first argument too. Then we will say that the bracket $[.,.]$ defines a skew-symmetric bidifferential operator on the associative commutative algebra $\mathcal{F}$. Consequently we will define abstractly algebras of smooth functions on Jacobi manifolds as follows [Gr92].

Definition 2. A Jacobi algebra is an associative commutative algebra A with unit equipped with a skew-symmetric bidifferential operator P which defines a Lie algebra structure on A.

It can be shown that if A contains no non-trivial nilpotent elements then the differential operator P is of order $\leqslant 1$ and then we have the following alternative formulation of theorem 1 [Gr92].

Theorem 2. Let A be an associative commutative algebra with unit containing no non-zero nilpotent elements, then a skew-symmetric differential operator P defines a Jacobi structure on A iff there exists a bivector Λ and a vector field X verifying (i) and (ii) in definition 1, such that $P(f, g) = [f, g]$ where $[.,.]$ is the bracket defined by (2).

Even if Jacobi algebras are not Poisson algebras in general, we can define Hamiltonian vector fields as follows:

Proposition 1. The map $\natural : \mathcal{F}(M) \to \mathfrak{X}(M)$ defined by

$$f \mapsto \natural(f) = X_f = \Lambda(\mathrm{d}f) + fX \qquad \forall f \in \mathcal{F}(M)$$

is a Lie algebra homomorphism, i.e.,

$$[X_f, X_g] = X_{[f,g]}.$$

The vector fields X_f will be called Hamiltonian vector fields and f will be called the Hamiltonian of X_f.

We shall denote by $\mathcal{F}_X$ the subalgebra of functions invariant under X, i.e. $\mathcal{F}_X = \{f \in \mathcal{F} | \mathcal{L}_X f = 0\}$ whose elements will be called basic functions on M. If the vector field X is fibrating, i.e. the space of integral curves N defines a regular foliation of M such that the canonical projection $\pi : M \to N$ is a submersion, it is obvious that $\mathcal{F}_X = \pi^*(\mathcal{F}(N))$

where $\mathcal{F}(N)$ denotes the algebra of smooth functions on the manifold N [Ju84]. The triple (M, Λ, X) has been called a regular Jacobi manifold and Λ projects onto a Poisson bivector Λ_N on N [Ch96].

2.2. Derivations of the Lie algebra structure

A vector field Y on M is a derivation of the Lie algebra structure on $\mathcal{F}$ defined by a Jacobi structure (Λ, X) if by definition,

$$\mathcal{L}_Y[f, g] = [\mathcal{L}_Y f, g] + [f, \mathcal{L}_Y g] \qquad \forall f, g \in \mathcal{F}. \tag{4}$$

Notice that in addition Y is a derivation of the associative structure of $\mathcal{F}$. This is the geometrical translation of the notion of a derivation of a Jacobi algebra. Such derivations can be called Jacobi derivations. It is an immediate computation to check that X is a Jacobi derivation. Then, it follows:

Proposition 2. The subalgebra $\mathcal{F}_X$ is a Lie subalgebra of $(\mathcal{F}(M), [.,.])$.

If Y is a derivation of the Lie algebra $[.,.]$, by using $f =$ constant in the previous formula (4) we get

$$\mathcal{L}_Y[f, g] = [f, \mathcal{L}_Y g]$$

but using the definition (2), $[f, g] = f\mathcal{L}_X g$, then

$$\mathcal{L}_Y(\mathcal{L}_X g) = \mathcal{L}_X(\mathcal{L}_Y g) \qquad \forall g \in \mathcal{F}$$

then

$$[X, Y] = 0.$$

By using now, $f, g \in \mathcal{F}_X$, we find

$$\mathcal{L}_Y \Lambda = 0.$$

Thus we have proved that Y is an infinitesimal automorphism of the Jacobi structure (Λ, X). In fact, the converse is also true as can be checked directly and we get,

Proposition 3. The vector field Y is a derivation for $[.,.]$ iff $\mathcal{L}_Y X = 0$ and $\mathcal{L}_Y \Lambda = 0$.

Hence, propositions 2 and 3 imply the following.

Corollary 1. The Lie algebra structure $[.,.]$ induces a Poisson structure on $\mathcal{F}_X$.

Thus, the formal quotient manifold whose algebra of functions is $\mathcal{F}_X$, is a Poisson manifold. If X is fibrating then the quotient space N is actually a Poisson manifold as was indicated earlier. In this sense we can say that Jacobi manifolds are extensions of Poisson manifolds. These remarks provide a third way of characterizing Jacobi algebras. It is obvious that the constant functions on M form an Abelian subalgebra of $\mathcal{F}$. Moreover, it is clear that the centralizer of the Abelian subalgebra of constant functions is $\mathcal{F}_X$. Thus a Jacobi algebra can be characterized as a unital algebra A such that the operator $X = D_{1_A}$ is a derivation and the centralizer of the Abelian subalgebra defined by multiples of the unit element 1_A is a Poisson algebra and it coincides with the invariant subalgebra of X.

Hamiltonian vector fields are not Jacobi derivations in general, however we have, *Proposition 4.* A Hamiltonian vector field X_h is an (inner) derivation iff $\mathcal{L}_X h = 0$, i.e. $h \in \mathcal{F}_X$.

Definition 3. A function C on M will be said to be a Casimir function if $[C, f] = 0$ for all $f \in \mathcal{F}$.

Notice that $[C, f] = 0$, $\forall f$, implies that

$$\mathcal{L}_X C = 0$$

because by definition, equation (2),

$$[C, f] = \Lambda(\mathrm{d}C, \mathrm{d}f) + C\mathcal{L}_X f - f\mathcal{L}_X C$$

hence using $f =$ constant, we obtain the above conclusion. Besides the arbitrariness of f implies that

$$\Lambda(\mathrm{d}C, .) = 0$$

and then for any Hamiltonian vector field

$$\mathcal{L}_{X_f} C = 0.$$

The converse follows the same argument, then we have,

Proposition 5. A function C is a Casimir iff $\mathcal{L}_{X_f} C = 0$ for all $f \in \mathcal{F}$.

2.3. *Particular cases*

Let (M, Λ, X) be a Jacobi manifold. We will describe several particular cases for Λ and X that lead to some well known geometrical structures.

(I) $X = 0$. Then (M, Λ) becomes a Poisson manifold. If Λ is non-degenerate we obtain a symplectic manifold with symplectic structure ω the inverse of the tensor Λ. The Lie bracket $[., .]$ becomes the ordinary Poisson bracket defined by the symplectic form ω.

(II) $X = 0$. Let (M, Ω, η) be a co-symplectic manifold, say, Ω is a closed 2-form and η is a closed 1-form on M such that $\Omega^n \wedge \eta \neq 0$, with $\dim M = 2n + 1$. There exists a Reeb vector field ξ defined by the equations $i_\xi \Omega = 0$, $i_\xi \eta = 1$, but M carries a Poisson structure (see [Ca92], [Le93], [Ch96]). We can define a map $\flat(X) = i_X \Omega + (i_X \eta)\eta$ from vector fields to 1-forms. Then we can define a Poisson tensor Λ as

$$\Lambda(\alpha, \beta) = \Omega(\flat^{-1}(\alpha), \flat^{-1}(\beta)) \tag{5}$$

for all covectors α, β on M.

A Jacobi manifold of constant rank k is defined by the condition $X \wedge \Lambda^k \neq 0$ and $\Lambda^{k+1} = 0$. Jacobi manifolds of maximal rank are such that $X \wedge \Lambda^n \neq 0$ with $\dim M = 2n+1$, then there exists a 1-form θ such that $i_\theta \Lambda = X$ and θ is a contact 1-form. Conversely we have,

(III) $X \neq 0$. Let M be a contact manifold with contact form θ, i.e. $\theta \wedge \mathrm{d}\theta^n \neq 0$. Then let X be the Reeb field of (M, θ), $i_X \theta = 1$, $i_X\, \mathrm{d}\theta = 0$. Let $\flat$ be the map defined similarly to the case of co-symplectic manifolds, this is, $\flat(X) = i_X\, \mathrm{d}\theta + (i_X \theta)\theta$. Then Λ is defined as in (5) replacing Ω by $\mathrm{d}\theta$, but now (Λ, X) define a Jacobi structure. The bracket $[., .]$ becomes the Lagrange bracket for contact structures [Lb58] (see the example A in section 2.4).

The Jacobi structure defined on any contact manifold can also be constructed in some situations as follows. If the quotient space N of M by the Reeb field is a manifold, it inherits a symplectic structure ω and θ defines a connection on the fibration $M \to N$. We lift horizontally the inverse of the symplectic structure on N by using the connection θ, obtaining Λ in this way.

(IV) $\Lambda = 0$. In this case we get a Witt algebra. For instance on $M = S^1$, we find a Virasoro algebra as a central extension of it [Gr96].

2.4. Some examples

2.4.1. The Jacobi structure on $T_0^* SU(2)$. We will consider first the manifold $SU(2)$. We realize $SU(2)$ in terms of matrices g of the form

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \qquad \alpha, \beta \in \mathbb{C}$$

satisfying $|\alpha|^2 + |\beta|^2 = 1$. We define the left invariant vector fields X_1^L, X_2^L, X_3^L to be given by the equations,

$$i_{X_a^L} \Theta_L = \mathrm{i}\sigma_a$$

where σ_1, σ_2, σ_3 are Pauli matrices and Θ_L the left-invariant Maurer–Cartan form $g^{-1}\,\mathrm{d}g$.

We set

$$\Lambda = X_1^L \wedge X_2^L \qquad X = X_3^L. \tag{6}$$

Clearly X defines the Hopf fibration $\pi : S^3 \to S^2$ and $\mathcal{F}_X = \pi^*(\mathcal{F}(S^2))$. It is trivial to check that

$$[\Lambda, \Lambda] = 2X \wedge \Lambda \qquad [X, \Lambda] = 0$$

using the commutation properties of the vector field X_a^L given by

$$[X_a^L, X_b^L] = \epsilon_{abc} X_c^L.$$

Then the previous tensors Λ, X define a Jacobi structure on the manifold S^3 which is the Jacobi structure defined by the contact structure defined on S^3 by the contact 1-form $\theta = -\frac{1}{2}\mathrm{i}\,\mathrm{Tr}(\sigma_3 \Theta_L)$.

It is simple to show that the previous Jacobi structure, when restricted to S^2, gives the standard Poisson structure on S^2. Indeed, Λ is invariant under left translations on the group $SU(2)$.

Because $X = X_3^L$ is a generator of the right action, the left action projects onto an action on S^2. The Poisson bracket induced on $\mathcal{F}_X$ is invariant under this action. On S^2 all $SO(3)$ invariant bivector fields are multiple of each other.

Now we consider the cotangent group of $SU(2)$ without the zero section. We will denote it by $T_0^* SU(2)$. We will identify it with the product $SU(2) \times \mathbb{R}_0^3$, where $\mathbb{R}_0^3 = \mathbb{R}^3 - \{\mathbf{0}\}$, by using left translations again. Then

$$T_0^* SU(2) \cong SU(2) \times \mathbb{R}_0^3. \tag{7}$$

We will consider the constant rank Poisson structure on $\mathbb{R}_0^3$ obtained from the identification $\mathbb{R}_0^3 \cong S^2 \times \mathbb{R}^+$. We will denote this Poisson tensor by Λ_2 and it coincides again with the canonical linear Poisson structure on the dual of the Lie algebra of $SU(2)$. Then we define the following objects,

$$\Lambda = \Lambda_1 + \frac{\partial}{\partial r} \wedge X_1 - \mathrm{e}^r \Lambda_2 \qquad X = X_1 \tag{8}$$

where Λ_1, X_1 denotes the Jacobi structure on $SU(2)$ defined above, (6), and r denotes the radial coordinate in $\mathbb{R}_0^3$. We must point out that the Jacobi structure thus defined is a non-trivial superposition of a contact and a Poisson structure. In fact this Jacobi manifold is non-transitive, i.e. $\natural(T^*M) + \langle X \rangle$ does not span TM. It is well known that Jacobi structures obtained from contact and/or co-symplectic manifolds must be transitive. Thus, the Jacobi structure defined above does not fall into any of the previous geometric categories.

2.4.2. Locally conformal symplectic manifolds. Another interesting example of Jacobi manifolds is provided by the class of locally conformal symplectic manifolds. Such manifolds are characterized by a non-degenerate 2-form Φ and a closed 1-form ω, called the Lee form of the structure, verifying (see [Co86] and references therein),

$$\mathrm{d}\Phi = \omega \wedge \Phi. \tag{9}$$

We can define a map $\flat$ sending vectors into 1-forms by contraction with Φ. Then the tensors $\Lambda = \Phi \circ \flat^{-1}$ and $X = \flat^{-1}(\omega)$ define a Jacobi structure.

There are abundant examples of genuine locally conformal symplectic manifolds. We will simply quote the following [Co86]. Let $H(r, 1)$ denote the Heisenberg group of dimension $2r + 1$. We will denote its coordinates by q^i, p_i, $i = 1, \ldots, r$, and s. We will now consider the extension $H(r, 1) \times \mathbb{R}$ with the extra coordinate denoted by t. Then we define on this manifold the 2-form,

$$\Phi = \mathrm{d}q^i \wedge \mathrm{d}p_i + (\mathrm{d}s - p_i\,\mathrm{d}q^i) \wedge \mathrm{d}t.$$

It is now obvious that $\mathrm{d}\Phi = \mathrm{d}t \wedge \Phi$ and Φ defines a locally conformal symplectic structure with Lee form $\mathrm{d}t$. The Jacobi structure defined by the locally conformal symplectic structure $(\Phi, \mathrm{d}t)$ is given by the tensors:

$$\Lambda = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial s} - p_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial s} \qquad X = \frac{\partial}{\partial s}. \tag{10}$$

We will not address here the mechanical interpretation of this structure but we will consider the quotient of this manifold (diffeomorphic to $\mathbb{R}^{2r+2}$ by the natural right action of the subgroup $\Gamma(r, 1) \times \mathbb{Z}$ of $H(r, 1) \times R$) given by those elements with integer coordinates. It is a simple computation to show that Φ, $\mathrm{d}t$ are invariant with respect to the action of the discrete subgroup $\Gamma(r, 1) \times \mathbb{Z}$, hence it passes to the quotient space $N(r) \times S^1 = (H(r, 1) \times \mathbb{R})/(\Gamma(r, 1) \times \mathbb{Z})$ inducing on this compact nilmanifold a locally conformal structure with very peculiar characteristics. For instance it is known that $N(r) \times S^1$ can have no Kähler structures, and if $r \geqslant 2$ then it cannot have symplectic structures either [Co86].

Other examples of locally conformal symplectic manifolds obtained by quotienting nilpotent or solvable groups with remarkable properties are discussed for instance in [Fe88], [An88].

3. Generalized reduction of Jacobi manifolds

3.1. Reduction of commutative associative algebras

The generalized reduction process can be described better in the algebraic setting of commutative associative algebras. This way of thinking exploits the duality among topological spaces and the correspondings algebras of functions defined on them.

The idea is to obtain a reduced algebra out of a given one A combining two elementary processes:

—First, 'choosing a submanifold', by definition this consists in choosing a quotient B of the algebra A. The projection map $\pi : A \to B$ defines an ideal I as $\ker \pi = I$ and $A/I = B$. Thus, 'choosing a submanifold' is equivalent to fixing an ideal I of A.

—The second process, 'quotienting out an equivalence relation', by definition consists in choosing a subalgebra A_I of A/I. We will call the algebra A_I a reduction of the algebra A.

Notice that there are two choices involved in the definition of the reduced algebra A_I, the choice of I and of the subalgebra A_I itself. This can be made more explicit as follows. The inverse image of the subalgebra A_I defines a subalgebra $A' = \pi^{-1}(A_I)$ of A. Thus, the order in which the two processes are performed can be reversed realizing that $A_I = A'/A' \cap I$. Then we can first select a subalgebra A' of A and then an ideal I' of A' and define $A_I = A'/I'$. The relation between the ideal I and I' is that I is the ideal generated by I', i.e. $I = AI'$. We see in this way the equivalence between the two ways of constructing the reduced algebra A_I and the dependence of A_I in the choice of a subalgebra A'.

It is obvious that the reduction process can be repeated a number of times using, each time, an ideal I_a of the reduced algebra $A_{I_{a-1}}$ and a subalgebra A_{I_a} of the quotient algebra $A_{I_{a-1}}/I_a$, $a = 1, \ldots, n$, with $A_{I_0} = A_I$. It is obvious from the previous remark that we can reorganize the ideals I_a and the subalgebras A_{I_a} to obtain a unique ideal J such that A_{I_n} is a subalgebra of A/J. Thus the reduction process can always be restricted to the two steps described above.

The previous ideas can be refined if we suppose that the algebra A carries some additional structure. For instance we can suppose that A carries a Lie algebra structure $[.,.]$ (not necessarily compatible with the associative product). Then if we are given an ideal I as before (with respect to the associative structure), we would like to choose the subalgebra A_I of A/I such that it will inherit a Lie algebra structure, reproducing in this way the structures in the original algebra A. Let us assume that A' is simultaneously a Lie subalgebra of $(A, [.,.])$ and a subalgebra of $(A, \cdot)$. Now we will suppose that $A'_I \cap I$ is an ideal with respect to the associative structure and an invariant Lie subalgebra of A', then it is obvious that the quotient algebra $A_I = A'/A' \cap I$ is a subalgebra of A/I and carries a Lie algebra structure. Thus the reduced algebra A_I can also be called a reduced Lie algebra for A.

From the above considerations it is obvious that the conditions imposed on the subalgebra A' are very tight. This shows that the enormous freedom we have to construct reduced algebras is only apparent. We will show in the coming section how to find adequate subalgebras A' for Jacobi algebras.

Before that we will briefly analyse the reduction of dynamics. By definition, dynamical systems on algebras are derivations of the algebra. Thus it is obvious that if D is a derivation of an associative commutative algebra A and we want to reduce both the algebra and the dynamics, the subalgebra A' and the ideal I must be invariant with respect to D, i.e. $D(I) \subset I$, $D(A') \subset A'$. In this case it is obvious that D induces a derivation D_I on the reduced algebra A_I that will be called the reduced dynamic induced by D on A_I.

3.2. Reduction of Jacobi manifolds

We will apply the ideas in the previous section to the algebra of smooth functions on a Jacobi manifold M. Thus, the above algebra A will now be $\mathcal{F}$. We will choose an ideal of the associative commutative algebra $\mathcal{F}$ that will be denoted now as $\mathcal{J}$, i.e. $\mathcal{F}\mathcal{J} \subset \mathcal{J}$ and we shall assume that $\mathcal{L}_X \mathcal{J} \subset \mathcal{J}$, i.e. $\mathcal{J}$ is X-invariant.

With the ideal $\mathcal{J}$ we can associate as before the short exact sequence of associative commutative algebras

$$0 \to \mathcal{J} \to \mathcal{F} \to \mathcal{F}/\mathcal{J} \to 0.$$

The ideal $\mathcal{J}$ allows us to choose directly the subalgebra A' used to complete the reduction process. This subalgebra is the normalizer of $\mathcal{J}$ with respect to the Lie algebra structure $[.,.]$ on $\mathcal{F}$ which is defined as

$$\mathcal{N}_{\mathcal{J}} = \{f \in \mathcal{F} | [f, \mathcal{J}] \subset \mathcal{J}\}. \tag{11}$$

The following propositions are devoted to show that $\mathcal{N}_{\mathcal{J}}$ verifies all the properties required in the reduction process. We must remark about the crucial role played by the invariance of $\mathcal{J}$ with respect to X in the proof of some of them. In particular this means that the reduction process as discussed here will not work for an arbitrary Lie algebra structure on an associative commutative algebra.

Proposition 6. $\mathcal{N}_{\mathcal{J}}$ is a Lie subalgebra of $(\mathcal{F}, [.,.])$.

Proof. Is an immediate consequence of the Jacobi identity for $[.,.]$, because for $f, g \in \mathcal{N}_{\mathcal{J}}$,

$$[[f, g], \mathcal{J}] = [[g, \mathcal{J}], f] + [[\mathcal{J}, f], g] \tag{12}$$

the term inside the first bracket on the right-hand side of the previous equation (12), is in $\mathcal{J}$ because $g \in \mathcal{N}_{\mathcal{J}}$. Then because $f \in \mathcal{N}_{\mathcal{J}}$ we obtain that the first term is in $\mathcal{J}$. The same argument applies to the second term in the right-hand side of (12). □

Proposition 7. The subalgebra $\mathcal{N}_{\mathcal{J}}$ is X-invariant, i.e. $\mathcal{L}_X \mathcal{N}_{\mathcal{J}} \subset \mathcal{N}_{\mathcal{J}}$.

Proof. Let f be a function in $\mathcal{N}_{\mathcal{J}}$ and h a function in $\mathcal{J}$. Because X is a derivation of $[.,.]$, we have,

$$\mathcal{L}_X[f, h] = [\mathcal{L}_X f, h] + [f, \mathcal{L}_X h].$$

Then,

$$[\mathcal{L}_X f, h] = \mathcal{L}_X[f, h] - [f, \mathcal{L}_X h]$$

is in $\mathcal{J}$ because $[f, h]$ and $\mathcal{L}_X h$ are in $\mathcal{J}$. □

Proposition 8. The subspace $\mathcal{N}_{\mathcal{J}}$ is a subalgebra of $\mathcal{F}$ with respect to the associative commutative structure $\cdot$.

Proof. Let h be a function on $\mathcal{J}$ and f_1, f_2 two arbitrary functions on $\mathcal{N}_{\mathcal{J}}$. Then,

$$[f_1 f_2, h] = f_1[f_2, h] + f_2[f_1, h] - f_1 f_2 \mathcal{L}_X h.$$

The two first terms on the right-hand side of the previous equation belong to $\mathcal{J}$ because f_i are in $\mathcal{N}_{\mathcal{J}}$, and the last term is also contained in $\mathcal{J}$ because $\mathcal{J}$ is X-invariant and is an ideal, hence $f_1 f_2$ is in $\mathcal{N}_{\mathcal{J}}$. □

Proposition 9. $\mathcal{N}_{\mathcal{J}} \cap \mathcal{J}$ is an invariant Lie subalgebra of $\mathcal{N}_{\mathcal{J}}$, i.e.

$$[\mathcal{N}_{\mathcal{J}} \cap \mathcal{J}, \mathcal{N}_{\mathcal{J}}] \subset \mathcal{N}_{\mathcal{J}} \cap \mathcal{J}.$$

Proof. Let f be in $\mathcal{N}_{\mathcal{J}} \cap \mathcal{J}$ and g in $\mathcal{N}_{\mathcal{J}}$. Then f is in $\mathcal{N}_{\mathcal{J}}$ and then proposition 6 implies that $[f, g] \in \mathcal{N}_{\mathcal{J}}$, but $[f, g]$ is also in $\mathcal{J}$ because of the definition of the normalizer $\mathcal{N}_{\mathcal{J}}$ (11). Then, $[f, g] \in \mathcal{N}_{\mathcal{J}} \cap \mathcal{J}$.

Now if h_1, h_2 are in $\mathcal{N}_{\mathcal{J}} \cap \mathcal{J}$, then $[h_1, h_2]$ is in $\mathcal{N}_{\mathcal{J}}$ because h_1, h_2 are in $\mathcal{N}_{\mathcal{J}}$ and proposition 6. On the other hand $h_2 \in \mathcal{J}$, but $h_1 \in \mathcal{N}_{\mathcal{J}}$, then $[h_1, h_2] \in \mathcal{J}$. Hence $[h_1, h_2] \in \mathcal{N}_{\mathcal{J}} \cap \mathcal{J}$. □

Proposition 10. The subspace $\mathcal{N}_{\mathcal{J}} \cap \mathcal{J}$ is an ideal of the associative commutative algebra $\mathcal{N}_{\mathcal{J}}$.

Proof. Let f be in $\mathcal{N}_{\mathcal{J}} \cap \mathcal{J}$ and $g \in \mathcal{N}_{\mathcal{J}}$. Then, $f \in \mathcal{J}$ and $gf \in \mathcal{J}$ because $\mathcal{J}$ is an ideal. Moreover, proposition 8 implies that $\mathcal{N}_{\mathcal{J}}$ is a subalgebra of $\mathcal{F}$ with respect to $\cdot$, then $gf \in \mathcal{N}_{\mathcal{J}}$ and the conclusion follows. □

Thus we can state the following theorem.

Theorem 3. Let (M, Λ, X) be a Jacobi manifold and $\mathcal{J}$ an ideal of the associative commutative algebra of smooth functions $\mathcal{F}$ on M. Let us suppose that $\mathcal{J}$ is X-invariant. Then, the quotient space $\mathcal{N}_{\mathcal{J}}/\mathcal{N}_{\mathcal{J}} \cap \mathcal{J}$ inherits a Jacobi algebra structure induced from that of $\mathcal{F}$. Moreover, if there is a smooth manifold R such that $\mathcal{F}(R) = \mathcal{N}_{\mathcal{J}}/\mathcal{N}_{\mathcal{J}} \cap \mathcal{J}$, then R inherits the structure of a Jacobi manifold, the bracket among functions given by the bracket of the Lie algebra structure induced in $\mathcal{N}_{\mathcal{J}}/\mathcal{N}_{\mathcal{J}} \cap \mathcal{J}$.

The algebra $\mathcal{F}_{\mathcal{J}} = \mathcal{N}_{\mathcal{J}}/\mathcal{N}_{\mathcal{J}} \cap \mathcal{J}$ will be called the reduced algebra of the Jacobi algebra $\mathcal{F}$ with respect to the ideal $\mathcal{J}$ and the Lie bracket on it will be denoted by $[.,.]_{\mathcal{J}}$. If the associative commutative structure on $\mathcal{F}_{\mathcal{J}} = \mathcal{N}_{\mathcal{J}}/\mathcal{N}_{\mathcal{J}} \cap \mathcal{J}$ defines a smooth manifold R, the reduced structure defined on this quotient manifold R will be called the reduction of the Jacobi manifold M.

Proof. We see that because of proposition 9, the Lie algebra (proposition 6) $\mathcal{N}_{\mathcal{J}}$ admits $\mathcal{N}_{\mathcal{J}} \cap \mathcal{J}$ as an invariant Lie subalgebra, therefore $\mathcal{N}_{\mathcal{J}}/\mathcal{N}_{\mathcal{J}} \cap \mathcal{J}$ is a Lie algebra. Moreover, because of proposition 10, $\mathcal{N}_{\mathcal{J}} \cap \mathcal{J}$ is an ideal of $\mathcal{N}_{\mathcal{J}}$ with respect to its associative structure (proposition 8) and the quotient $\mathcal{N}_{\mathcal{J}}/\mathcal{N}_{\mathcal{J}} \cap \mathcal{J}$ is an associative commutative algebra. Finally the derivation X passes to the quotient because of proposition 7. Thus we have on the reduced algebra $\mathcal{N}_{\mathcal{J}}/\mathcal{N}_{\mathcal{J}} \cap \mathcal{J}$ an associative and a Lie algebra structure. It remains to show that they define a Jacobi algebra. This follows from the following lemma.

Lemma 1. Let D be a differential operator of order r on the associative commutative algebra A. If I is an ideal such that $D(I) \subset I$ then, the linear operator $\bar{D}$ induced by D on the quotient A/I is again a differential operator of order $\leqslant r$.

Proof. The operator $\bar{D}(x + I) = D(x) + I$ defined on A/I verifies that

$$\delta(\bar{x})\bar{D} = \overline{\delta(x)D}$$

for all $x \in \bar{x} = x + I$. Then,

$$\delta(\bar{x})^{r+1}\bar{D} = \overline{\delta(x)^{r+1}D} = 0$$

because $\delta(x)^{r+1}D = 0$ (D is of order r). □

Then we conclude the proof of the main theorem, noting that the operator D_f given by (1), leaves invariant $\mathcal{N}_{\mathcal{J}}$ for all $f \in \mathcal{N}_{\mathcal{J}}$ (proposition 6), then D_f is a differential operator of order 1 in $\mathcal{N}_{\mathcal{J}}$ for all $f \in \mathcal{N}_{\mathcal{J}}$. Moreover, $\mathcal{J}$ is invariant for each D_f, $f \in \mathcal{N}_{\mathcal{J}}$. Thus the operator D_f induces a differential operator $\overline{D}_f$ of order 1 in the quotient algebra $\mathcal{N}_{\mathcal{J}}/\mathcal{N}_{\mathcal{J}} \cap \mathcal{J}$. By definition of the Lie bracket in the reduced algebra, we have,

$$\overline{D}_f(\bar{g}) = [\bar{f}, \bar{g}]_{\mathcal{J}}.$$

Thus the Lie bracket $[., .]_{\mathcal{J}}$ defines a skew-bidifferential operator of order 1, hence a Jacobi structure. Finally, if the reduced algebra is the algebra of functions of a smooth manifold R, theorem 2 implies that R is a Jacobi manifold. □

Remark. It is important to remark that there is not a simple criterion to recognize if the reduced algebra $\mathcal{F}_{\mathcal{J}}$ is the algebra of functions of a smooth manifold. An alternative road could be taken considering the reduction of topological algebras (for instance C^*-algebras). In such a case it is clear that with the obvious modifications the previous theorem still works. Nevertheless, we must realize that this does not help us to obtain a geometrical interpretation of Jacobi manifolds because the smooth structure will be obtained by selecting a subalgebra on the algebra of functions which is not provided by the theorem. In some particular situations, however, it is possible to prove that there exists a reduced manifold. Essentially all these situations use a slicing theorem for group actions (see for instance section 4.2).

3.3. Reduction by a submanifold

The canonical way of defining ideals in spaces of functions is by fixing subspaces. Let Σ be an embedded closed submanifold of M and we denote by $\mathcal{J}_\Sigma$ the ideal of smooth functions vanishing on Σ, $\mathcal{J}_\Sigma = \{f \in \mathcal{F} | f|_\Sigma = 0\}$. Then $\mathcal{F}(\Sigma) \cong \mathcal{F}(M)/\mathcal{J}_\Sigma$.

It is clear that $\mathcal{J}_\Sigma$ is X-invariant iff Σ is invariant under the flow of X or equivalently, if $X|_\Sigma \in T\Sigma$. The normalizer $\mathcal{N}_\Sigma$ of $\mathcal{J}_\Sigma$ consists of functions f such that the operator D_f leaves $\mathcal{J}_\Sigma$ invariant. It is simple to check that this is equivalent to asking whether the Hamiltonian vector field X_f is tangential to Σ. Recall that because of proposition 7, X also induces a derivation of the normalizer $\mathcal{N}_\Sigma$.

Some terminology is convenient now. Functions in $\mathcal{J}_\Sigma$ will be called constraint functions and functions in $\mathcal{N}_\Sigma$ will be called first-class functions. Functions in $\mathcal{N}_\Sigma \cap \mathcal{J}_\Sigma$ will be called first-class constraints and constraints which are not in $\mathcal{N}_\Sigma \cap \mathcal{J}_\Sigma$ will be called second-class constraints. First-class constraints define Hamiltonian vector fields which are tangential to Σ but because of proposition 1 and proposition 9 we have that the distribution $\mathcal{D}_\Sigma$ generated by Hamiltonian vector fields corresponding to first-class constraints is an integrable distribution. It defines a foliation of M whose restriction to the submanifold Σ will be denoted by $\mathcal{L}_\Sigma$.

In general the reduced Jacobi algebra will not be the algebra of smooth functions on the quotient space $\Sigma/\mathcal{L}_\Sigma$ unless some further conditions are imposed on Σ. For instance if Σ is such that $\mathcal{J}_\Sigma \subset \mathcal{N}_\Sigma$, we will say that Σ is first class or co-isotropic. Then if Σ is X-invariant, the reduced Jacobi algebra is $\mathcal{N}_\Sigma/\mathcal{J}_\Sigma$. It is not hard to see that in this case $\mathcal{F}(\Sigma/\mathcal{L}_\Sigma) = \mathcal{N}_\Sigma/\mathcal{J}_\Sigma$ and the space of leaves of the foliation $\mathcal{L}_\Sigma$ inherits a Jacobi structure. To conclude we can state that if the foliation is regular and Σ is co-isotropic, then the reduced Jacobi algebra is the algebra of functions of the quotient manifold $\Sigma/\mathcal{L}_\Sigma$.

Simple cases of co-isotropic submanifolds of Jacobi manifolds are provided by the following examples.

Let S be a regular level set of the Casimir functions on M. Then it is clear that $\mathcal{J}_S$ is the ideal generated by the subalgebra of Casimirs on M. Then, $\mathcal{N}_S = \mathcal{F}$ because $[f, C] = 0$ for all $f \in \mathcal{F}$, and C an arbitrary Casimir. Hence, $\mathcal{J}_S \subset \mathcal{N}_S = \mathcal{F}$ and the submanifold S is co-isotropic. In this particular case, the reduced algebra is $\mathcal{F}/\mathcal{J}_S \cong \mathcal{F}(S)$ and the reduced manifold is S itself with the natural Jacobi structure induced on it by Λ and X.

Another example is provided by submanifolds S such that $\mathcal{J}_S + \mathcal{N}_S = \mathcal{F}$, then because $\mathcal{N}_S/\mathcal{N}_S \cap \mathcal{J}_S \cong \mathcal{N}_S + \mathcal{J}_S/\mathcal{J}_S$, then, the reduced algebra will again be $\mathcal{F}/\mathcal{J}_S = \mathcal{F}(S)$ and the reduced manifold will be the submanifold S again that will inherit a Jacobi structure. In particular we will obtain as corollaries of this situation most of the results in [Da91], for instance:

Theorem 4. Let S be a submanifold of the Jacobi manifold M such that it is X-invariant and satisfies,

$$TS + \Lambda^\sharp(TS^0) = TM$$

where $T_xS^0 = \{\alpha \in T_x^*M | \alpha(v) = 0, \forall v \in T_xS\}$ denotes the polar distribution to TS and $\Lambda^\sharp$ is the bundle map $T^*M \to TM$ defined by contraction with Λ. Then S inherits in a natural way the structure of a Jacobi manifold.

3.4. Reduction by a distribution

It is natural to select as a subalgebra in the reduction process not just the normalizer of an ideal $\mathcal{J}$ but a subalgebra of it selected by imposing some invariance requirement. This is formulated using integrable distributions on $\mathcal{F}$, i.e. Lie subalgebras $\mathcal{D}$ of the Lie algebra of derivations of the algebra $\mathcal{F}$. Any distribution defines an associative subalgebra of invariant functions, $\mathcal{F}_\mathcal{D} = \{f \in \mathcal{F} | Z(f) = 0, \forall Z \in \mathcal{D}\}$. We will say that the integrable distribution $\mathcal{D}$ is compatible with the Jacobi algebra structure of $\mathcal{F}$ if $\mathcal{F}_\mathcal{D}$ is an X-invariant subalgebra with respect to both structures, associative and Lie algebra, on $\mathcal{F}$.

It is now obvious that if $\mathcal{J}$ is an ideal on $\mathcal{F}$, the quotient algebra

$$\mathcal{F}_{\mathcal{J},\mathcal{D}} = \mathcal{F}_\mathcal{D} \cap \mathcal{N}_\mathcal{J} / \mathcal{F}_\mathcal{D} \cap \mathcal{N}_\mathcal{J} \cap \mathcal{J}$$

is a Jacobi algebra, that will be called the reduced Jacobi algebra of $\mathcal{F}$ with respect to $\mathcal{J}$ and $\mathcal{D}$. The results are evident from the previous discussion in that if $\mathcal{J} = 0$, then the reduced Jacobi algebra is just $\mathcal{F}_\mathcal{D}$ which amounts to performing only step 2, 'quotienting by an equivalence relation', in the reduction process. One particular instance of this situation arises when we have a submersion $\pi : M \to N$. Then the tangent spaces to the level sets of π define a distribution $\mathcal{D}$ and the algebra $\mathcal{F}_\mathcal{D}$ is precisely the algebra generated by functions of the form $f \circ \pi$, $f \in \mathcal{F}(N)$. We will see two examples of this situation in the next section.

4. Examples and applications

4.1. Reduction of the Jacobi manifolds $T_0^*SU(2)$ and $J^1(Q, \mathbb{R}) \times \mathbb{R}$

Let us consider the Jacobi structure defined in section 2.4. Then there are two natural projections, $\pi : T_0^*SU(2) \to SU(2)$ given by the natural projection on the first factor in the decomposition given by (7), and $J : T_0^*SU(2) \to \mathbb{R}_0^3$ given by the projection onto the second factor. Each one of the maps π, J defines distributions on $T_0^*SU(2)$ given by the vector fields tangent to the corresponding level sets. It is clear that the subalgebras of invariant functions are generated by the components of the maps π and J themselves, and because of (8) it is clear that they are X-invariant Lie subalgebras of the full algebra $\mathcal{F}(T_0^* SU(2))$ of functions. Thus the contact structure on $SU(2)$ and the Lie–Poisson structure on $\mathfrak{su}(2)$ are obtained by generalized reduction of the Jacobi structure of $T_0^* SU(2)$.

We can also consider the locally conformal symplectic structure defined in section 3.4.2. The Jacobi structure (Λ, X) on $\mathbb{R}^{2r+2}$ is a particular example of the following situation. Let Q be a smooth manifold and $J^1(Q, \mathbb{R}) \times \mathbb{R} \cong T^*Q \times \mathbb{R} \times \mathbb{R}$ with local coordinates q^i, p_i, s, t. This manifold posseses a canonical Jacobi structure with tensors given locally by (10). Again we have two natural projections $\pi : J^1(Q, \mathbb{R}) \times \mathbb{R} \to J^1(Q, \mathbb{R})$ and $\rho : J^1(Q, \mathbb{R}) \times \mathbb{R} \to T^*Q \times \mathbb{R}$. The spaces $J^1(Q, \mathbb{R})$ and $T^*Q \times \mathbb{R}$ have respectively a contact and a co-symplectic structure. It is a simple check to show that they are precisely the reduction of the Jacobi structure in the total space by the corresponding projections. In this sense we can think of the Jacobi structure in $J^1(Q, \mathbb{R}) \times \mathbb{R}$ as a non-trivial mixing of contact and co-symplectic structures.

We will consider as a further example the reduction of $H(r, 1) \times \mathbb{R}$ with the Jacobi structure given by (10) with respect to some submanifolds. For instance, consider the submanifold defined by the equation $q^i = 0$ and the ideal $\mathcal{J}$ defining the submanifold is the ideal generated by q^i. We can easily compute the Jacobi brackets among the generators q^i, p_i, s, t of the algebra of functions on $H(r, 1) \times \mathbb{R}$ using (2),

$$
\begin{array}{llll}
[q^i, q^j] = 0 & [q^i, p_j] = \delta^i_j & [q^i, s] = q^i & [q^i, t] = 0 \\
[p_i, p_j] = 0 & [p_i, s] = 0 & [p_i, t] = 0 & [s, t] = t - 1.
\end{array}
$$

Then the normalizer $\mathcal{N}_{\mathcal{J}}$ of $\mathcal{J}$ is generated by $q^1, p_1, \ldots, q^i, q^{i+1}, p_{i+1}, \ldots, q^r, p_r, s, t$. The submanifold is co-isotropic and the reduced algebra is generated by $q^1, p_1, \ldots, q^{i-1}, p_{i-1}, q^{i+1}, p_{i+1}, \ldots, q^r, p_r, s, t$, i.e it is the algebra of functions of $H(r-1, 1) \times \mathbb{R}$ with its Jacobi structure.

Let us now consider the submanifold $t = 0$. Then the normalizer of the ideal generated by t is generated by q^i, p_i, t and the submanifold is co-isotropic. The reduced algebra is generated by q^i, p_i. Thus it coincides with $\mathbb{R}^{2r}$ and the induced Jacobi structure is the canonical symplectic structure on it.

4.2. Reduction of Jacobi manifolds with symmetry

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.

Definition 4. A Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(M)$, $a \mapsto \xi_a$, defines an infinitesimal action of G on the Jacobi manifold M by automorphisms if ξ_a is a derivation for the Lie bracket on $\mathcal{F}$. The action of $\mathfrak{g}$ on M will be called Hamiltonian if ξ_a is Hamiltonian for all $a \in \mathfrak{g}$.

Notice that the infinitesimal action of $\mathfrak{g}$ on M is Jacobian iff $\mathcal{L}_{\xi_a}\Lambda = 0$, $[\xi_a, X] = 0$, for all $a \in \mathfrak{g}$. We also notice that an action by infinitesimal automorphisms is Hamiltonian if the Hamiltonians f_a associated with the elements of $\mathfrak{g}$ are in $\mathcal{F}_X$. Such an action will be called Jacobian.

We will consider in what follows a Jacobian action of a Lie group on a Jacobi manifold M. Denoting as before the Hamiltonian defined by the element $a \in \mathfrak{g}$ by f_a we have,

$$\xi_a = X_{f_a}.$$

Hence we define a momentum map $\mu : M \to \mathfrak{g}^*$ by setting

$$\langle \mu, a \rangle = f_a \qquad \forall a \in \mathfrak{g} \tag{13}$$

or equivalently,

$$\xi_a = \langle \mu, a\rangle X + \Lambda(d\langle \mu, a\rangle) \qquad \forall a \in \mathfrak{g}. \tag{14}$$

If $\mathbf{0}$ is a weakly regular value for the momentum map μ, then $\Sigma = \mu^{-1}(\mathbf{0})$ is a submanifold of M. Consider the ideal $\mathcal{J}_\Sigma$ of functions vanishing on Σ. It is obvious that $\mathcal{J}_\Sigma$ is the ideal on $\mathcal{F}$ generated by the functions f_a, $a \in \mathfrak{g}$ and consequently it will be denoted by $\mathcal{J}_0$.

We will denote by $\mathcal{N}_0$ the normalizer of $\mathcal{J}_0$. It is not hard to check that Σ is first class, i.e. $\mathcal{J}_0 \subset \mathcal{N}_0$. Moreover, $\mathcal{N}_0$ is the algebra of functions such that their restriction to Σ are G-invariant, i.e. if $\mathcal{D}$ denotes the integrable distribution generated by ξ_a, $a \in \mathfrak{g}$, then $\mathcal{N}_0 \subset \mathcal{F}_{\mathcal{D}}$. The reduced algebra will therefore be $\mathcal{F}_0 = \mathcal{N}_0/\mathcal{J}_0$ or equivalently the algebra of functions on Σ invariant with respect to $\mathcal{D}$.

If the quotient space $\mu^{-1}(\mathbf{0})/G$ is a smooth manifold, for instance this will be the case if G is a compact Lie group acting properly on M, then the quotient algebra $\mathcal{F}_0$ can be identified with the algebra of functions on Σ/G. Then, the Jacobi reduction theorem allows us to conclude that the quotient manifold Σ/G inherits the structure of a Jacobi manifold.

4.3. *Reduction of symplectic, Poisson, contact and co-symplectic structures*

4.3.1. Reduction of symplectic manifolds. $X = 0$, Λ non-degenerate. The previous procedure agrees with symplectic reduction as discussed for instance in [Ma85], [Gr94]. In fact, specializing the discussion in section 4.2 to symplectic manifolds with symmetry we will obtain the well known Marsden–Weinstein symplectic reduction theorem. Notice that an action of a group G is Jacobian if $i_{\xi_a}\omega = \mathrm{d}f_a$, where ω is the symplectic form defined by Λ. Thus the momentum map μ defined by (13) and (14) coincides with the symplectic momentum map. Then finally the reduced algebra $\mathcal{F}_0$ coincides with the algebra of functions of $\mu^{-1}(\mathbf{0})/G$ (provided that it actually defines a smooth manifold).

4.3.2. Reduction of Poisson manifolds. $X = 0$, Λ arbitrary. The previous discussion leads us to the construction of reduced Poisson manifolds as discussed for instance in [Gr94].

If we are given a submanifold Σ of a Poisson manifold M and a subbundle E of TM, we can define the annihilator E^0_Σ of E restricted to Σ, then we consider the distribution $\mathcal{D}$ generated by Hamiltonian vector fields such that the differentials of their Hamiltonians lie on E^0_Σ. If E verifies the conditions stated in [Ma86], then the distribution $\mathcal{D}$ is compatible with the Poisson structure and the reduced Poisson agrees with the Marsden–Ratiu reduction of M by Σ and E.

4.3.3. Reduction of co-symplectic manifolds. $X = 0$. Particularizing the results above to co-symplectic manifolds we will obtain the reduction of co-symplectic manifolds with symmetry [Al89]. The reduction of co-symplectic manifolds for singular momentum maps discussed in [Le93] can be described in this setting with the obvious modifications.

4.3.4. Reduction of contact manifolds. $X \neq 0$. Similarly, reduction of contact manifolds with symmetry (see for instance [Al89], [Le96]) is included in our previous discussion.

Note added in proof. It was called to our attention that two previous papers have dealt with reduction of Jacobi manifolds, however from a geometric perspective: Margarida J and Nunes da Costa M 1989 Réduction des variétés de Jacobi *C. R. Acad. Sci., Paris* I **308** 101–3; and Mikami K 1989 Reduction of local Lie algebras *Proc. Am. Math. Soc.* **105** 686. A forthcoming work will discuss the relationship between them and this paper.

Acknowledgments

The author AI wishes to acknowledge the partial financial support provided by DGICYT under the programme PB92-0197 as well as the NATO collaborative research grant 940195. The author ML acknowledges the partical finantial support provided by DGICYT project PB94-0106.

References

[Al89] Albert C 1989 Le théorème de réduction de Marsden–Weinstein en géometrie cosymplectique et de contact *J. Geom. Phys.* **6** 627–49

[An88] de Andrés L C, Fernández M, Mencia J J and Rozas J I 1988 Four-dimensional compact locally conformal Kähler solvmanifolds *Boll. U.M.I.* A **2** 377–82

[Ca90] Cantrijn F and Ibort A 1990 Reduction of Poisson supermanifolds *C. R. Acad. Sci., Paris* **311** 567–72

[Ca92] Cantrijn F, de León M and Lacomba E A 1992 Gradient vector fields on cosymplectic manifolds *J. Phys. A: Math. Gen.* **25** 175–88

[Ch96] Chinea D, de León M and Marrero J C 1996 A canonical differential complex for Jacobi manifolds *Preprint* IMAFF–CSIC

[Co86] Cordero L A, Fernández M and de León M 1986 Compact locally conformal Kähler nilmanifolds *Geom. Dedicata* **21** 187–92

[Da91] Dazord P, Lichnerowicz A and Marle C-M 1991 Structure locale des variétés de Jacobi *J. Math. Pures Appl.* **70** 101–52

[Fe88] Fernández M, Gotay M J and Gray A 1988 Four-dimensional parallelizable symplectic and complex manifolds *Proc. Am. Math. Soc.* **103** 1203–12

[Ge94] Getzler E 1994 Batalin–Vilkovisky algebras and two-dimensional topological field theories *Commun. Math. Phys.* **159** 265–2865

[Gr92] Grabowski J 1992 Abstract Jacobi and Poisson structures. Quantization and star-products *J. Geom. Phys.* **9** 45–73

[Gr94] Grabowski J, Landi G, Marmo G and Vilasi G 1994 Generalized reduction procedure: symplectic and Poisson formalism *Forts. Phys.* **42** 393–427

[Gr96] Grabowski J, Marmo G, Perelomov A and Simoni A 1996 Remarks on Virasoro and Kac–Moody algebras *Preprint*

[Gu84] Guedira F and Lichnerowicz A 1984 Géometrie des algèbres de Lie locales de Kirillov *J. Math. Pures Appl.* **63** 407–84

[Ju84] Justino A M 1984 Propiétés du quotient d'une variété de Jacobi pour une feuilletage engendré par des automorphismes infinitésimaux *C. R. Acad. Sci., Paris* I **298** 489–92

[Ki93] Kimura T 1993 Generalized classical cohomology and reduction of Poisson manifolds. *Commun. Math. Phys.* **151** 155–82

[Ki76] Kirillov A 1976 Local Lie algebras *Russ. Math. Surv.* **31** 55–75

[Le93] de León M and Saralegui M 1993 Cosymplectic reduction for singular momentum maps *J. Phys. A: Math. Gen.* **26** 5033–43

[Le96] de León M, Marrero J C and Tuynman G 1996 $\mathbb{R}^{2n+1}$ is a universal contact manifold for reduction *Preprint*

[Lb58] Libermann P 1958 Sur les automorphismes infinitésimaux des structures symplectiques et de contact *Colloque de Géométrie Differentielle Globale (Bruxelles)* pp 19–22

[Li78] Lichnerowicz A 1978 Les variétés de Jacobi et leurs algèbres de Lie associeés *J. Math. Pures Appl.* **57** 453–88

[Ma85] Marmo G, Saletan E J, Simoni A and Vitale B 1985 *Dynamical Systems, a Differential Geometric Approach to Symmetry and Reduction* (New York: Wiley)

[Ma96] Marmo G and Ibort A 1996 A generalized reduction procedure for dynamical systems *Proc. IV Fall Workshop on Differential Geometry and its Applications (Santiago de Compostela 95). (RSEF, Anales de Física, Monografías 3*) pp 55–70

[Mr96] Marmo G, Vilasi G and Vinogradov A 1996 In preparation

[Ma74] Marsden J E and Weinstein A 1974 Reduction of symplectic manifolds with symmetry *Rep. Math. Phys.* **5** 121–30

[Ma86] Marsden J and Ratiu T 1986 Reduction of Poisson manifolds *Lett. Math. Phys.* **11** 161–9

[Sc54] Schouten J A 1954 *Conv. Int. Geom. Diff. Cremonese ed. Roma*

[Sc95] Kosmann-Schwarzbach Y 1995 From Poisson algebras to Gerstenhaber algebras *Centre de Mathématiques, Ecole Polytechnique Preprint* 1122

[Sh74] Shiga K 1974 Cohomology of Lie algebras over a manifold I, II *J. Math. Soc. Japan* **26** 324–61

[Tu74] Tulczyjew W 1974 The graded Lie algebra of multivector fields and the generalized derivative of forms *Bull. Acad. Pol. Sci.* **22** 937–42

[Wi90] Witten E 1990 A note on the antibracket formalism *Mod. Phys. Lett.* A **5** 487–94